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On Asynchronous Stochastic Automata

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The paper deals with generalized stochastic automata (probabilistic sequential machines) which are able to print not only a single output but an output tape of arbitrary finite length each unit of time. Section 1 contains the basic definitions and some observations concerning the basic probabilities which are associated with each stochastic automaton. In section 2 the equivalence of situations and of automata is investigated. The principal result is the decidability of equivalence of finite situations of finite automata. Section 3, the main portion of the paper, is devoted to an investigation of the input-output relations, i.e., of the externally observable behavior, of stochastic automata. The existence of a so-called state family is shown to be a criterion for the generability of a stochastic operator within a stochastic automaton. The nonuniqueness of state families leads to a consideration of separability of stochastic operators, a property being, for generable stochastic operators, necessary and sufficient for the uniqueness of a state family. This property as well as a somewhat weaker property is characterized then. Some open problems are pointed out.

INTRODUCTION

In this paper a generalization of the usual definition of stochastic automata (probabilistic sequential machines) is presented. This generalization concerns the output of the automata: The stochastic automata considered here are allowed to print not only one output within each unit of time but an output tape of arbitrary finite length. In general, if such an automaton has worked a finite number of moments, say $t + 1, \ldots, t + n$, it is impossible to dissect its output tape into subtapes corresponding to the single moments, i.e., it is impossible to recognize which subtape of its output tape has been printed at time $t + i$ $(1 \leqslant i \leqslant n)$. From this fact problems arise, which convinced us of the importance of our concept. Moreover, the theory developed here seems to be of interest not only to automata theorists but to workers in coding and communication theory too.

643/17/3-4

1. Basic Definitions

Let A be an arbitrary nonempty set. By $W(A)$ we denote the set of all finite sequences (called tapes) $p = a_1a_2 \cdots a_n$ of elements a_i from A and enclude in $W(A)$ the empty tape e. The length of a tape $p \in W(A)$ is denoted by $l(p)$. A tape p is called a prefix of the tape q (denoted by $p \sqsubseteq q$) if for some tape r the equation $pr = q$ holds. In case $r \neq e$ p is a proper prefix of q (denoted by $p \sqsubset q$). A prefix p of q is said to be a n-prefix of q, if p is of length n.

A discrete probability measure over A is a realvalued function P from the set $\mathfrak{P}(A)$ of all subsets of A such that

(K1) $P(E) \geqslant 0$ for each $E \in \mathfrak{P}(A)$,

(K2) $P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$ for each sequence $E_1, E_2, \ldots$ of pairwise disjoint subsets E_i of A,

(K3) $P(A) = 1$, and there exists an utmost countable subset E of A such that $P(E) = 1$.

The probability measure P is discrete in the sense that the set

$$A^+ =_{\mathrm{df}} \{a \mid a \in A \wedge P(\{a\}) > 0\}$$

is at most countable albeit A may be a noncountable set. If no confusion is possible we write $P(a)$ instead of $P(\{a\})$. The set of all discrete probability measures over A is denoted by Π_A.

For shortness we use the notation of predicate calculus.

Definition 1.1. An (asynchronous) stochastic automaton is a 4-tuple $\mathfrak{C} = [X, Y, Z, H]$ where X, Y and Z are nonempty sets and H is a function $[z, x] \to H[z, x]$ from $Z \times X$ into $\Pi_{W(Y) \times Z}$.

The set X is the set of inputs x of $\mathfrak{C}$, Y is the set of outputs y and Z is the set of states of $\mathfrak{C}$. The interpretation of a stochastic automaton is as a system working within a discrete time scale as follows. If $z \in Z$ is the state and $x \in X$ is the input at time t then with probability $H[z, x](q, z')$ the system will produce the output tape $q \in W(Y)$ at time t and will be in state $z' \in Z$ at time $t + 1$. Thus only input and state transition are synchronized. Each input x causes exactly one state transition, but there may be states z and inputs x such that if the system is in state z and receives input x the output unit will produce nothing (i.e., will produce the empty tape e) or a tape $q \in W(Y)$ of length greater than 1. Hence in general the length of the output tape differs from the length of a corresponding input tape.

DEFINITION 1.2. Let be $\mathfrak{C} = [X, Y, Z, H]$ a stochastic automaton.

1. $\mathfrak{C}$ is said to be synchronous if for each $z, z' \in Z, x \in X, q \in W(Y)$ from $H[z, x](q, z') > 0$ it follows $l(q) = 1$ (i.e. $q \in Y$).

2. $\mathfrak{C}$ is called weakly finite, if X, Y and Z are finite sets and finite, if moreover the set

$$W_{\mathfrak{C}} =_{\mathrm{df}} \{q \mid \exists z \exists z' \exists x (z, z' \in Z \wedge x \in X \wedge H[z, x](q, z') > 0\}$$

is finite.

3. The discrete probability measures $\mathfrak{y} \in \Pi_{W(Y) \times Z}$ over $W(Y) \times Z$ are called (stochastic) situations of $\mathfrak{C}$ and the elements $\mathfrak{z} \in \Pi_Z$ stochastic states of $\mathfrak{C}$.

4. A situation is deterministic if there exists a pair $[s, z] \in W(Y) \times Z$ such that $\mathfrak{y}(s, z) = 1$ and is, in this case, denoted by $\delta_{s,z}$. Analogously stochastic states $\mathfrak{z}$ with $\mathfrak{z}(z) = 1$ for some $z \in Z$ will be called deterministic and denoted by δ_z.

5. A situation $\mathfrak{y} \in \Pi_{W(Y) \times Z}$ of $\mathfrak{C}$ is said to be finite if the set

$$\{s \mid s \in W(Y) \wedge \mathfrak{y}(\{s\} \times Z) > 0\}$$

is finite.

Clearly synchronous automata are exactly the ones investigated by Carlyle and others since 1961. Obviously $W_{\mathfrak{C}}$ is the set of all output tapes which $\mathfrak{C}$ is able to print with nonzero probability within one unit of time. Let be $\mathfrak{y}$ a situation of $\mathfrak{C}$. The interpretation of the real number $\mathfrak{y}(s, z)$ ($s \in W(Y), z \in Z$) is as the probability that the automaton $\mathfrak{C}$ has entered the state z and has produced the output tape s before starting its work (i.e., before receiving the first input).

If $\mathfrak{y}$ is a situation of $\mathfrak{C}$ with

$$\forall s \forall z (s \in W(Y) \wedge z \in Z \wedge \mathfrak{y}(s, z) > 0 \rightarrow s = e) \qquad (*)$$

to $\mathfrak{y}$ there corresponds biuniquely the stochastic state $\mathfrak{z} \in \Pi_Z$ defined by

$$\mathfrak{z}(z) =_{\mathrm{df}} \mathfrak{y}(e, z) \qquad \text{for each } z \in Z.$$

In what follows we sometimes identify stochastic states with the corresponding situations (having the property (*)) on the one hand and states $z \in Z$ with the corresponding deterministic stochastic states δ_z and situations $\delta_{e,z}$ on the other hand.

It is obvious that many of the definitions and facts known from the theory of synchronous stochastic automata (cf., e.g., Carlyle (1969) and Starke (1969)) may be reformulated for the asynchronous case; in general we omit the details.

Let be $\mathfrak{C} = [X, Y, Z, H]$ an arbitrary stochastic automaton. We now extend the domain of the function H and define the function $V_{\mathfrak{C}}$ describing the external behavior of the automaton $\mathfrak{C}$.

For $z_1, z_{n+1} \in Z$, $q \in W(Y)$, $p = x_1 \cdots x_n \in W(X)$ we define

$$H[z_1, p](q, z_{n+1}) =_{\mathrm{df}} \begin{cases} 1, & \text{if } \; p = e \quad (\text{i.e., } n = 0), \quad q = e, \quad z_1 = z_{n+1}, \\ \displaystyle\sum_{\substack{q_1 \cdots q_n = q \\ q_1, \ldots, q_n \in W(Y)}} \; \sum_{z_2, \ldots, z_n \in Z} \; \prod_{i=1}^{n} H[z_i, x_i](q_i, z_{i+1}), & \text{if } \; n > 0, \\ 0, & \text{else.} \end{cases}$$

Thus $H[z, p](q, z')$ is the probability that $\mathfrak{C}$ will be in state z' at time $t + l(p)$ and produce the output tape q within the time interval $(t, t + l(p) - 1)$ if $\mathfrak{C}$ is started in the state z at time t and has received the input tape p within the time interval $(t, t + l(p) - 1)$.

For $q, s \in W(Y)$, $p \in W(X)$ and $\mathfrak{y} \in \Pi_{W(Y) \times Z}$ let be

$$\frac{q}{s} =_{\mathrm{df}} \{w \mid sw = q\},$$

$$H[\mathfrak{y}, p](q, z') =_{\mathrm{df}} \sum_{\substack{s \in W(Y) \\ z \in Z}} \mathfrak{y}(s, z) H[z, p] \left(\frac{q}{s} \times \{z'\}\right)$$

and

$$V_{\mathfrak{C}}[\mathfrak{y}, p](q) =_{\mathrm{df}} H[\mathfrak{y}, p](\{q\} \times Z).$$

Obviously $V_{\mathfrak{C}}[\mathfrak{y}, p](q)$ is the probability that the output tape q will be produced, if $\mathfrak{C}$ starts in situation $\mathfrak{y}$ and receives the input tape p, i.e., the function $V_{\mathfrak{C}}$ describes the external behavior of the stochastic automaton $\mathfrak{C}$.

THEOREM 1.1. *Let $\mathfrak{C} = [X, Y, Z, H]$ be an arbitrary stochastic automaton and $\mathfrak{y} \in \Pi_{W(Y) \times Z}$, $p, r \in W(X)$, $w \in W(Y)$, $z' \in Z$.*

1. $H[\mathfrak{y}, e](w, z') = \mathfrak{y}(w, z')$ (i.e., $H[\mathfrak{y}, e] = \mathfrak{y}$).
2. $V_{\mathfrak{C}}[\mathfrak{y}, e](w) = \mathfrak{y}(\{w\} \times Z)$.
3. $H[\mathfrak{y}, p] \in \Pi_{W(Y) \times Z} \wedge V_{\mathfrak{C}}[\mathfrak{y}, p] \in \Pi_{W(Y)}$.
4. $H[\mathfrak{y}, pr] = H[H[\mathfrak{y}, p], r]$.
5. $V_{\mathfrak{C}}[\mathfrak{y}, pr] = V_{\mathfrak{C}}[H[\mathfrak{y}, p], r]$.

Proof. The assertions 1.1.1 and 1.1.2 are easily verified. To prove 1.1.3 it suffices to show that $H[\mathfrak{y}, p](W(Y) \times Z) = 1$ holds. We have

$$H[\mathfrak{y}, p](W(Y) \times Z) = \sum_{\substack{w \in W(Y) \\ z \in Z}} \sum_{\substack{s \in W(Y) \\ z_1 \in Z}} \mathfrak{y}(s, z_1) H[z_1, p] \left(\frac{w}{s} \times \{z\}\right)$$

$$= \sum_{\substack{s \in W(Y) \\ z_1 \in Z}} \mathfrak{y}(s, z_1) \sum_{\substack{w \in W(Y) \\ z \in Z}} H[z_1, p] \left(\frac{w}{s} \times \{z\}\right).$$

Now $H[z_1, p](w/s \times \{z\}) > 0$ implies $w/s \neq \emptyset$, i.e., w is of form sq. From this follows

$$\sum_{\substack{w \in W(Y) \\ z \in Z}} H[z_1, p] \left(\frac{w}{s} \times \{z\}\right) = \sum_{\substack{w \in \{s\} W(Y) \\ z \in Z}} H[z_1, p] \left(\frac{w}{s} \times \{z\}\right)$$

$$= H[z_1, p](W(Y) \times Z) = 1,$$

which implies

$$H[\mathfrak{y}, p](W(Y) \times Z) = \sum_{\substack{s \in W(Y) \\ z_1 \in Z}} \mathfrak{y}(s, z_1) \cdot 1 = 1.$$

We prove now the assertion 1.1.4 which implies 1.1.5.

$$H[\mathfrak{y}, pr](w, z')$$

$$\sum_{\substack{s \in W(Y) \\ z_1 \in Z}} \mathfrak{y}(s, z_1) H[z_1, pr] \left(\frac{w}{s} \times \{z'\}\right)$$

$$= \sum_{\substack{s \in W(Y) \\ z_1 \in Z}} \mathfrak{y}(s, z_1) \sum_{\substack{q \in W(Y) \\ z \in Z}} H[z_1, p](q, z) H[z, r] \left(\frac{w}{sq} \times \{z'\}\right)$$

$$= \sum_{\substack{v \in W(Y) \\ z \in Z}} \sum_{\substack{s \in W(Y) \\ z_1 \in Z}} \mathfrak{y}(s, z_1) H[z_1, p] \left(\frac{v}{s} \times \{z\}\right) H[z, r] \left(\frac{w}{v} \times \{z'\}\right)$$

$$= \sum_{\substack{v \in W(Y) \\ z \in Z}} H[\mathfrak{y}, p](v, z) H[z, r] \left(\frac{w}{v} \times \{z'\}\right)$$

$$= H[H[\mathfrak{y}, p], r](w, z').$$

2. Equivalence

In this section the equivalence of situations (and as a consequence that of states and stochastic states) of stochastic automata and moreover the equivalence and the weak equivalence of stochastic automata is defined. Some important theorems known for synchronous automata are established for the asynchronous case. The main result is the decidability of equivalence of finite situations of finite asynchronous automata. From the results of this section it is easily seen, that it is possible to reduce and minimize asynchronous stochastic automata just in the same manner as in the synchronous case. Therefore minimization problems are not considered here.

Definition 2.1. Let be $\mathfrak{C} = [X, Y, Z, H]$ and $\mathfrak{C}' = [X, Y, Z', H']$ stochastic automata having the same set of inputs and the same set of outputs.

1. Situations $\mathfrak{y} \in \Pi_{W(Y)\times Z}$, $\mathfrak{y}' \in \Pi_{W(Y)\times Z'}$ of $\mathfrak{C}$, $\mathfrak{C}'$ respectively are said to be equivalent (denoted by $\mathfrak{y} \sim \mathfrak{y}'$) if the discrete probability measures $V_{\mathfrak{C}}[\mathfrak{y}, p]$, $V_{\mathfrak{C}'}[\mathfrak{y}', p]$ are equal for all $p \in W(X)$.

2. The automaton $\mathfrak{C}$ is called equivalently embedded in $\mathfrak{C}'$ (denoted by $\mathfrak{C} \underset{\sim}{\subseteq} \mathfrak{C}'$) if to each state of $\mathfrak{C}$ there corresponds an equivalent state of $\mathfrak{C}'$, and weakly equivalently embedded in $\mathfrak{C}'$ (denoted by $\mathfrak{C} \underset{\approx}{\subseteq} \mathfrak{C}'$), if to each situation of $\mathfrak{C}$ there corresponds an equivalent situation of $\mathfrak{C}'$.

3. The automata $\mathfrak{C}$, $\mathfrak{C}'$ are said to be equivalent (denoted by $\mathfrak{C} \sim \mathfrak{C}'$) if $\mathfrak{C} \underset{\sim}{\subseteq} \mathfrak{C}'$ and $\mathfrak{C}' \underset{\sim}{\subseteq} \mathfrak{C}$, and weakly equivalent (denoted by $\mathfrak{C} \approx \mathfrak{C}'$) if $\mathfrak{C} \underset{\approx}{\subseteq} \mathfrak{C}'$ and $\mathfrak{C}' \underset{\approx}{\subseteq} \mathfrak{C}$.

In the theory of synchronous stochastic automata an automaton $\mathfrak{C}$ is said to be weakly equivalently embedded in an automaton $\mathfrak{C}'$ if to each stochastic state of $\mathfrak{C}$ there corresponds an equivalent stochastic state of $\mathfrak{C}'$. The definition above seems to be sharper, but the following theorem shows, that this is not the case.

Theorem 2.1. *Let be $\mathfrak{C}$ and $\mathfrak{C}'$ as in Definition 2.1. The automaton $\mathfrak{C}$ is weakly equivalently embedded in $\mathfrak{C}'$ if and only if to each state z of $\mathfrak{C}$ there corresponds an equivalent stochastic state $\mathfrak{z}_z'$ of $\mathfrak{C}'$.*

Proof. I. Let be $\mathfrak{C} \underset{\approx}{\subseteq} \mathfrak{C}'$ and $z \in Z$. To the state z there corresponds a situation $\mathfrak{y}'$ of $\mathfrak{C}'$ with $z \sim \mathfrak{y}'$. This implies by 1.1.2,

$$1 = V_{\mathfrak{C}}[z, e](e) = V_{\mathfrak{C}'}[\mathfrak{y}', e](e) = \mathfrak{y}'(\{e\} \times Z'),$$

i.e., $\mathfrak{y}'$ has the property (*). Thus the stochastic state $\mathfrak{z}' \in \Pi_{Z'}$ of $\mathfrak{C}'$ defined by

$$\mathfrak{z}'(z') =_{\mathrm{df}} \mathfrak{y}'(e, z')$$

(for each $z' \in Z'$) is equivalent with z.

II. To each $z \in Z$ let be fixed a stochastic state $\mathfrak{z}_z'$ of $\mathfrak{C}'$ such that $z \sim \mathfrak{z}_z'$, and let be $\mathfrak{y}$ an arbitrary situation of $\mathfrak{C}$. Define the situation $\mathfrak{y}'$ of $\mathfrak{C}'$ by

$$\mathfrak{y}'(s, z') =_{\mathrm{df}} \sum_{z \in Z} \mathfrak{y}(s, z) \cdot \mathfrak{z}_z'(z')$$

for all $s \in W(Y)$, $z' \in Z'$. We prove that $\mathfrak{y}$, $\mathfrak{y}'$ are equivalent. For $p \in W(X)$, $w \in W(Y)$

$$\begin{aligned} V_{\mathfrak{C}}[\mathfrak{y}, p](w) &= \sum_{\substack{s \in W(Y) \\ z \in Z}} \mathfrak{y}(s, z) V_{\mathfrak{C}}[z, p]\left(\frac{w}{s}\right) \\ &= \sum_{s \in W(Y)} \sum_{z \in Z} \mathfrak{y}(s, z) V_{\mathfrak{C}'}[\mathfrak{z}_z', p]\left(\frac{w}{s}\right) \qquad (z \sim \mathfrak{z}_z') \\ &= \sum_{s \in W(Y)} \sum_{z' \in Z} \sum_{z \in Z} \mathfrak{y}(s, z) \cdot \mathfrak{z}_z'(z') V_{\mathfrak{C}'}[z', p]\left(\frac{w}{s}\right) \\ &= V_{\mathfrak{C}'}[\mathfrak{y}', p](w). \end{aligned}$$

COROLLARY 2.2. 1. *If* $\mathfrak{C} \subsetsim \mathfrak{C}'$, *then* $\mathfrak{C} \underset{\approx}{\subset} \mathfrak{C}'$

2. *If* $\mathfrak{C} \sim \mathfrak{C}'$, *then* $\mathfrak{C} \approx \mathfrak{C}'$.

The converse of the assertions 2.2 do not hold even in the synchronous case (cf. Starke 1969).

In the same way as in the theory of synchronous stochastic automata one proves

THEOREM 2.3. *To each stochastic automaton* $\mathfrak{C}$ *there corresponds an equivalent stochastic Moore-automaton* $\mathfrak{C}'$ *being state-deterministic, weakly finite or finite if* $\mathfrak{C}$ *has the same property. Moreover to each stochastic automaton there corresponds an equivalent stochastic Moore-automaton* $\mathfrak{C}''$ *having a deterministic output function and being finite if* $\mathfrak{C}$ *is finite.*

(A stochastic automaton $[X, Y, Z, H]$ is a stochastic Moore-automaton if there exists an output function $M : z \to M[z]$ from Z into $\Pi_{W(Y)}$ such that the equation

$$H[z, x](s, z') = H[z, x](W(Y) \times \{z'\})\, M[z'](s)$$

is satisfied for all $z, z' \in Z$, $x \in X$, $s \in W(Y)$.)

Let be $\mathfrak{C} = [X, Y, Z, H]$ a stochastic automaton. For $\mathfrak{y} \in \Pi_{W(Y)\times Z}$, $p \in W(X)$, $q \in W(Y)$ with $V_{\mathfrak{C}}[\mathfrak{y}, p](q) > 0$ the stochastic state $\mathfrak{C}[\mathfrak{y}, p, q]$ of $\mathfrak{C}$ is defined by

$$\mathfrak{C}[\mathfrak{y}, p, q](z) =_{\mathrm{df}} \frac{H[\mathfrak{y}, p](q, z)}{V_{\mathfrak{C}}[\mathfrak{y}, p](q)}$$

for each $z \in Z$.

Corollary 2.4. *If* $\mathfrak{C} = [X, Y, Z, H]$ *is a stochastic automaton,* $\mathfrak{y} \in \Pi_{W(Y)\times Z}$, $p, r \in W(X)$, $w \in W(Y)$ *then*

$$V_{\mathfrak{C}}[\mathfrak{y}, pr](w) = \sum_{\substack{q \in W(Y) \\ V_{\mathfrak{C}}[\mathfrak{y}, p](q) > 0}} V_{\mathfrak{C}}[\mathfrak{y}, p](q) \cdot V_{\mathfrak{C}}[\mathfrak{C}[\mathfrak{y}, p, q], r]\left(\frac{w}{q}\right).$$

Theorem 2.5. *To each stochastic automaton* $\mathfrak{C} = [X, Y, Z, H]$ *there corresponds an observer-state-calculable stochastic automaton* $\mathfrak{C}^* = [X, Y, Z^*, H^*]$ *satisfying* $\mathfrak{C} \subseteq \mathfrak{C}^*$ *and* $\mathfrak{C} \approx \mathfrak{C}^*$.

Proof. Let be Z^* the least subset of Π_Z containing the deterministic stochastic states δ_z (for $z \in Z$) and closed under the operation of forming the stochastic states $\mathfrak{C}[\mathfrak{z}, x, s]$ for $\mathfrak{z} \in Z^*$, $x \in X$, $s \in W(Y)$. The function H^* is defined by

$$H^*[\mathfrak{z}, x](s, \mathfrak{z}') =_{\mathrm{df}} \begin{cases} V_{\mathfrak{C}}[\mathfrak{z}, x](s), & \text{if } V_{\mathfrak{C}}[\mathfrak{z}, x](s) > 0 \text{ and } \mathfrak{z}' = \mathfrak{C}[\mathfrak{z}, x, s] \\ 0, & \text{else.} \end{cases}$$

It is clear, that $\mathfrak{C}^*$ is observer-state-calculable, i.e., there exists a partial function δ^* from $Z^* \times X \times W(Y)$ into Z^* such that $H^*[\mathfrak{z}, x](s, \mathfrak{z}') > 0$ if and only if $\delta^*(\mathfrak{z}, x, s)$ is defined and $\mathfrak{z}' = \delta^*(\mathfrak{z}, x, s)$. One easily shows that each state $\mathfrak{z} \in Z^*$ of $\mathfrak{C}^*$ is equivalent with the (same) stochastic state $\mathfrak{z}$ of $\mathfrak{C}$ what proves Theorem 2.5.

Theorem 2.6. *If* $\mathfrak{C} = [X, Y, Z, H]$ *is a finite stochastic automaton with* n *states and if* $\mathfrak{z}, \mathfrak{z}'$ *are stochastic states and* $\mathfrak{y}, \mathfrak{y}'$ *finite situations of* $\mathfrak{C}$ *then*

1. $\mathfrak{z} \sim \mathfrak{z}' \leftrightarrow \forall p(p \in W(X) \wedge l(p) \leqslant n - 1 \to V_{\mathfrak{C}}[\mathfrak{z}, p] = V_{\mathfrak{C}}[\mathfrak{z}', p])$,
2. $\mathfrak{y} \sim \mathfrak{y}' \leftrightarrow \forall p(p \in W(X) \wedge l(p) \leqslant n \to V_{\mathfrak{C}}[\mathfrak{y}, p] = V_{\mathfrak{C}}[\mathfrak{y}', p])$.

Proof. In the synchronous theory the assertion 2.6.1 was discovered by Carlyle (1963). The proof presented here follows some of his ideas.

For $p \in W(X)$, $w \in W(Y)$ the real-valued function, which adjoins the number $V_{\mathfrak{C}}[z, p](w)$ with each $z \in Z$, is denoted by $V_{\mathfrak{C}}[\cdot, p](w)$. If M is a set of real-valued functions defined on Z then by $L(M)$ the linear space spanned by the functions in M is denoted, and by $\dim L(M)$ its dimension.
Let be

$$L =_{\mathrm{df}} L(\{V_{\mathfrak{C}}[\cdot, p](w) \mid p \in W(X) \wedge w \in W(Y)\})$$

and for $i = 0, 1, \ldots$

$$L_i =_{\mathrm{df}} L(\{V_{\mathfrak{C}}[\cdot, p](w) \mid p \in W(X) \wedge l(p) \leqslant i \wedge w \in W(Y)\}).$$

From this we have

$$L_0 \subseteqq L_1 \subseteqq L_2 \subseteqq \cdots \subseteqq L_i \subseteqq L_{i+1} \subseteqq \cdots \subseteqq L.$$

Obviously it is $\dim L_0 = 1$ and $\dim L \leqslant n$.
Since $\mathfrak{C}$ is finite, $W_{\mathfrak{C}}$ is finite. Let be

$$W_{\mathfrak{C}} =_{\mathrm{df}} \{s_1, \ldots, s_l\} \quad (l \geqslant 1).$$

LEMMA. *If* $L_i = L_{i+1}$ *then* $L_{i+1} = L_{i+2}$.

Proof. For $f \in L$ let be

$$I(f) =_{\mathrm{df}} \min\{i \mid f \in L_i\}.$$

We define a mapping T adjoining a subset $T(f)$ of L with each function $f \in L$. $T(f)$ is the set of all functions f' representable by a linear combination

$$f' = \sum_{\mu=1}^{m} \alpha_\mu \cdot V_{\mathfrak{C}}[\cdot, x_\mu p_\mu](q_\mu)$$

such that

$$f = \sum_{\mu=1}^{m} \alpha_\mu \cdot V_{\mathfrak{C}}[\cdot, p_\mu](q_\mu)$$

and the length of the tapes p_μ is at most $I(f)$, i.e.,

$$T(f) =_{\mathrm{df}} \left\{ \sum_{\mu=1}^{m} \alpha_\mu \cdot V_{\mathfrak{C}}[\cdot, x_\mu p_\mu](q_\mu) \;\middle|\; f = \sum_{\mu=1}^{m} \alpha_\mu \cdot V_{\mathfrak{C}}[\cdot, p_\mu](q_\mu) \right.$$

$$\left. \wedge \bigwedge_{\mu=1}^{m} l(p_\mu) \leqslant I(f) \wedge x_\mu \in X \right\}.$$

Moreover let be for $M \subseteqq L$

$$T(M) =_{\mathrm{df}} \bigcup_{f \in M} T(f).$$

First we prove that $T(L_j) = L_{j+1}$ for each $j = 0, 1, \ldots$.

If $f \in L_j$ then by definition of T each $f' \in T(f)$ is an element of $L_{I(f)+1} \subseteqq L_{j+1}$, since from $f \in L_j$ follows $I(f) \leqslant j$. Therefore we have $T(L_j) \subseteqq L_{j+1}$.

Let be f an arbitrary element of L_{j+1}, consequently $I(f) \leqslant j + 1$.

(i) $I(f) = 0$. Then $f \in L_0$, i.e., either f is identically zero or $f = \alpha V_{\mathfrak{C}}[\cdot, e](e)$ $(\alpha \neq 0)$. In the first case we consider the function $V_{\mathfrak{C}}[\cdot, e](q) \in L_0 \subseteqq L_j$ with $q \notin W_{\mathfrak{C}}$. Since $q \notin W_{\mathfrak{C}}$ for an arbitrarily chosen $x \in X$ we have

$$0 \equiv f = V_{\mathfrak{C}}[\cdot, x](q) \in T(V_{\mathfrak{C}}[\cdot, e](q)).$$

In the latter case we consider the function $\alpha \cdot \sum_{\lambda=1}^{l} V_{\mathfrak{C}}[\cdot, e](s_\lambda) \in L_0 \subseteqq L_j$ and obtain therefore

$$f = \alpha \cdot \sum_{\lambda=1}^{l} V_{\mathfrak{C}}[\cdot, x](s_\lambda) \in T\left(\alpha \cdot \sum_{\lambda=1}^{l} V_{\mathfrak{C}}[\cdot, e](s_\lambda)\right).$$

(ii) $0 < I(f) \leqslant j + 1$. Since $f \in L_{I(f)}$ to f there corresponds a linear combination

$$f = \sum_{\mu=1}^{m} \beta_\mu \cdot V_{\mathfrak{C}}[\cdot, p_\mu](q_\mu)$$

with $l(p_\mu) \leqslant I(f)$ for all $\mu = 1, \ldots, m$. Without loss of generality we can assume that there is at most one μ such that $p_\mu = e$, say p_m. If $q_m \neq e$ then $V_{\mathfrak{C}}[\cdot, p_m](q_m)$ is the zero-function and can be omitted.

In case that $p_m = q_m = e$ we choose an $x \in X$ and obtain

$$1 \equiv V_{\mathfrak{C}}[\cdot, p_m](q_m) = V_{\mathfrak{C}}[\cdot, e](e) = \sum_{\lambda=1}^{l} V_{\mathfrak{C}}[\cdot, x](s_\lambda) = V_{\mathfrak{C}}[\cdot, x](W_{\mathfrak{C}}).$$

Now let be

$$K =_{\mathrm{df}} m + l - 1,$$

$$\alpha_k =_{\mathrm{df}} \begin{cases} \beta_k & \text{if } 1 \leqslant k \leqslant m - 1 \\ \beta_m & \text{if } m \leqslant k \leqslant m + l - 1, \end{cases}$$

$$p_k' =_{\mathrm{df}} \begin{cases} p_k & \text{if } 1 \leqslant k \leqslant m - 1 \\ x & \text{if } m \leqslant k \leqslant m + l - 1, \end{cases}$$

$$q_k' =_{\mathrm{df}} \begin{cases} q_k & \text{if } 1 \leqslant k \leqslant m - 1 \\ s_{k-m+1} & \text{if } m \leqslant k \leqslant m + l - 1. \end{cases}$$

From this follows

$$f = \sum_{\mu=1}^{m-1} \beta_\mu \cdot V_{\mathfrak{C}}[\cdot, p_\mu](q_\mu) + \beta_m \cdot V_{\mathfrak{C}}[\cdot, e](e)$$

$$= \sum_{k=1}^{m-1} \alpha_k \cdot V_{\mathfrak{C}}[\cdot, p_k'](q_k') + \beta_m \cdot \sum_{k=m}^{m+l-1} V_{\mathfrak{C}}[\cdot, x](s_{k-m+1})$$

$$= \sum_{k=1}^{K} \alpha_k \cdot V_{\mathfrak{C}}[\cdot, p_k'](q_k').$$

Thus we have obtained a representation

$$f = \sum_{k=1}^{K} \alpha_k \cdot V_{\mathfrak{C}}[\cdot, p_k'](q_k')$$

of f satisfying

$$1 \leqslant l(p_k') \leqslant I(f) \qquad \text{for all } k = 1, \ldots, K.$$

Let be $x_k \in X$, $p_k'' \in W(X)$ such that

$$p_k' = x_k p_k'' .$$

Now we have

$$f = \sum_{k=1}^{K} \alpha_k \cdot V_{\mathfrak{C}}[\cdot, x_k p_k''](q_k') \in T\left(\sum_{k=1}^{K} \alpha_k \cdot V_{\mathfrak{C}}[\cdot, p_k''](q_k')\right)$$

with

$$\sum_{k=1}^{K} \alpha_k \cdot V_{\mathfrak{C}}[\cdot, p_k''](q_k') \in L_{I(f)-1} \subseteqq L_j$$

since

$$\bigwedge_{k=1}^{K} l(p_k'') \leqslant I(f) - 1.$$

Thus to each $f \in L_{j+1}$ there corresponds a $f' \in L_j$ such that

$$f \in T(f'), \qquad \text{i.e., } L_{j+1} \subseteqq T(L_j).$$

Now let be $L_i = L_{i+1}$. This implies

$$L_{i+1} = T(L_i) = T(L_{i+1}) = L_{i+2} . \qquad \text{Q.E.D.}$$

Let be $i^* =_{\text{df}} \min\{i \mid L_i = L_{i+1}\}$. From $1 \leqslant \dim L_i \leqslant n$ and

$$L_0 \subset L_1 \subset \cdots \subset L_{i^*-1} \subset L_{i^*} = L_{i^*+1} = L,$$

it follows $i^* \leqslant n - 1$ and $L_{n-1} = L$.

For stochastic states $\mathfrak{z}$, $\mathfrak{z}' \in \Pi_Z$ of $\mathfrak{C}$ and functions $f \in L$ we define

$$\Psi_{\mathfrak{z},\mathfrak{z}'}(f) =_{\mathrm{df}} \sum_{z\in Z} (\mathfrak{z}(z) - \mathfrak{z}'(z))f(z).$$

Obviously $\Psi_{\mathfrak{z},\mathfrak{z}'}$ is a linear functional on L. Therefore, we have

$$\begin{aligned}
\mathfrak{z} \sim \mathfrak{z}' &\leftrightarrow \forall p \forall q (p \in W(X) \wedge q \in W(Y) \rightarrow \Psi_{\mathfrak{z},\mathfrak{z}'}(V_{\mathfrak{C}}[\cdot, p](q)) = 0) \\
&\leftrightarrow \forall f (f \in L \rightarrow \Psi_{\mathfrak{z},\mathfrak{z}'}(f) = 0) \\
&\leftrightarrow \forall f (f \in L_{n-1} \rightarrow \Psi_{\mathfrak{z},\mathfrak{z}'}(f) = 0) \quad (\text{since } L_{n-1} = L) \\
&\leftrightarrow \forall p \forall q (p \in W(X) \wedge l(p) \leqslant n-1 \wedge q \in W(Y) \rightarrow \Psi_{\mathfrak{z},\mathfrak{z}'}(V_{\mathfrak{C}}[\cdot, p](q)) = 0) \\
&\leftrightarrow \forall p (p \in W(X) \wedge l(p) \leqslant n-1 \rightarrow V_{\mathfrak{C}}[\mathfrak{z}, p] = V_{\mathfrak{C}}[\mathfrak{z}', p]).
\end{aligned}$$

The assertion 2.6.1 is proved.

For proof of 2.6.2 let be $\mathfrak{y}$, $\mathfrak{y}' \in \Pi_{W(Y)\times Z}$ finite situations of $\mathfrak{C}$, z_0, $z_0' \notin Z$, $z_0 \neq z_0'$ and $Z^* =_{\mathrm{df}} Z \cup \{z_0, z_0'\}$.

We define H^* by

$$H^*[z^*, x](s, z^{**}) =_{\mathrm{df}} \begin{cases} H[z^*, x](s, z^{**}), & \text{if } \ z^*, z^{**} \in Z \\ \mathfrak{y}(s, z^{**}), & \text{if } \ z^* = z_0, \quad z^{**} \in Z \\ \mathfrak{y}'(s, z^{**}), & \text{if } \ z^* = z_0', \quad z^{**} \in Z \\ 0, & \text{else} \end{cases}$$

for z^*, $z^{**} \in Z^*$, $x \in X$, $s \in X$, $s \in W(Y)$. Let be $\mathfrak{C}^* =_{\mathrm{df}} [X, Y, Z^*, H^*]$. Since $\mathfrak{y}$, $\mathfrak{y}'$ are finite $\mathfrak{C}^*$ is a finite stochastic automaton with $n + 2$ states. Observe that

$$V_{\mathfrak{C}}[z, p] = V_{\mathfrak{C}^*}[z, p] \qquad \text{for all} \quad z \in Z, p \in W(X).$$

This implies

$$V_{\mathfrak{C}}[\mathfrak{y}, p] = V_{\mathfrak{C}^*}[z_0, xp] \quad \text{and} \quad V_{\mathfrak{C}^*}[\mathfrak{y}', p] = V_{\mathfrak{C}}[z_0', xp]$$

for all $p \in W(X)$ and each $x \in X$ by the following argument

$$\begin{aligned}
V_{\mathfrak{C}}[\mathfrak{y}, p](w) &= \sum_{\substack{s\in W(Y)\mathfrak{y} \\ z\in Z}} (s, z) V_{\mathfrak{C}}[z, p] \left(\frac{w}{s}\right) \\
&= \sum_{\substack{s\in W(Y) \\ z\in Z}} H^*[z_0, x](s, z) V_{\mathfrak{C}}[z, p] \left(\frac{w}{s}\right) \qquad (\text{for each } x \in X) \\
&= \sum_{\substack{s\in W(Y) \\ z\in Z^*}} H^*[z_0, x](s, z) V_{\mathfrak{C}^*}[z, p] \left(\frac{w}{s}\right) \\
&\qquad\qquad (\text{since } H^*[z_0, x](\{s\} \times \{z_0, z_0'\}) = 0) \\
&= V_{\mathfrak{C}^*}[z_0, xp](w).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\mathfrak{y} \sim \mathfrak{y}' &\leftrightarrow \forall p(p \in W(X) \to V_{\mathfrak{C}}[\mathfrak{y}, p] = V_{\mathfrak{C}}[\mathfrak{y}', p]) \\
&\leftrightarrow \forall p \forall x(p \in W(X) \wedge x \in X \to V_{\mathfrak{C}^*}[z_0, xp] = V_{\mathfrak{C}^*}[z_0', xp]) \\
&\leftrightarrow z_0 \sim z_0' \quad (\text{since } V_{\mathfrak{C}^*}[z_0, e] = V_{\mathfrak{C}^*}[z_0', e]) \\
&\leftrightarrow \forall p(p \in W(X) \wedge l(p) \leqslant n + 1 \to V_{\mathfrak{C}^*}[z_0, p] = V_{\mathfrak{C}^*}[z_0', p]) \\
&\hspace{20em} (\text{since } 2.6.1) \\
&\leftrightarrow \forall x \forall p(p \in W(X) \wedge x \in X \wedge l(p) \leqslant n \to V_{\mathfrak{C}^*}[z_0, xp] = V_{\mathfrak{C}^*}[z_0', xp]) \\
&\leftrightarrow \forall p(p \in W(X) \wedge l(p) \leqslant n \to V_{\mathfrak{C}}[\mathfrak{y}, p] = V_{\mathfrak{C}}[\mathfrak{y}', p]) \qquad \text{Q.E.D.}
\end{aligned}$$

COROLLARY 2.7. 1. *If $\mathfrak{y}$, $\mathfrak{y}'$ are finite situations of finite stochastic automata $\mathfrak{C}$, $\mathfrak{C}'$, respectively, one can effectively decide whether or not $\mathfrak{y} \sim \mathfrak{y}'$ holds.*

2. *If $\mathfrak{C}$, $\mathfrak{C}'$ are finite stochastic automata one can effectively decide whether or not $\mathfrak{C} \sim \mathfrak{C}'$ or $\mathfrak{C} \underset{\sim}{\subseteq} \mathfrak{C}'$ is satisfied.*

Many of the theorems known from the theory of synchronous stochastic automata, which refer to the equivalent or weakly equivalent simplification of stochastic automata, can be carried over without difficulties to the case considered here. Thereby one obtains, e.g., that for each finite stochastic automaton $\mathfrak{C}$ one can construct in an effective way a reduced form and a minimal form of $\mathfrak{C}$, that all minimal forms of the same stochastic automaton are pairwise equivalent, that one can effectively decide whether or not a finite stochastic automaton is in reduced form or in minimal form and whether or not $\mathfrak{C} \approx \mathfrak{C}'$ or $\mathfrak{C} \underset{\approx}{\subseteq} \mathfrak{C}'$ for finite stochastic automata $\mathfrak{C}$, $\mathfrak{C}'$ hold.

The following problem is left open.

PROBLEM 1. Let be $\mathfrak{C} = [X, Y, Z, H]$ a stochastic automaton satisfying

$$\forall \mathfrak{z} \forall \mathfrak{z}'(\mathfrak{z}, \mathfrak{z}' \in \Pi_Z \wedge \mathfrak{z} \sim \mathfrak{z}' \to \mathfrak{z} = \mathfrak{z}'). \tag{SR}$$

Does this imply

$$\forall \mathfrak{y} \forall \mathfrak{y}'(\mathfrak{y}, \mathfrak{y}' \in \Pi_{W(Y) \times Z} \wedge \mathfrak{y} \sim \mathfrak{y}' \to \mathfrak{y} = \mathfrak{y}')\,? \tag{VSR}$$

Let us call $\mathfrak{C}$ to be in strongly reduced form if condition (SR) is fulfilled and in very strongly reduced form if condition (VSR) holds. In the synchronous case the problem is solved by the following theorem.

THEOREM 2.8. *If a synchronous stochastic automaton $\mathfrak{C}$ is in strongly reduced form, it is in very strongly reduced form.*

Proof. Let be $\mathfrak{y}, \mathfrak{y}' \in \Pi_{W(Y)\times Z}$ with $\mathfrak{y} \sim \mathfrak{y}'$, and for $\mathfrak{y}'' \in \Pi_{W(Y)\times Z}$ define $W_{\mathfrak{y}''} = \{s \mid s \in W(Y) \wedge \mathfrak{y}''(\{s\} \times Z) > 0\}$. From $V_{\mathfrak{C}}[\mathfrak{y}', e] = V_{\mathfrak{C}}[\mathfrak{y}, e]$ we obtain for each $s \in W(Y)$

$$\mathfrak{y}(\{s\} \times Z) = V_{\mathfrak{C}}[\mathfrak{y}, e](s) = V_{\mathfrak{C}}[\mathfrak{y}', e](s) = \mathfrak{y}'(\{s\} \times Z),$$

i.e., $W_{\mathfrak{y}} = W_{\mathfrak{y}'}$. For each $s \in W_{\mathfrak{y}}$ consider the stochastic states $\mathfrak{z}_s$, $\mathfrak{z}_s'$ defined by

$$\mathfrak{z}_s(z) = \frac{\mathfrak{y}(s, z)}{\mathfrak{y}(\{s\} \times Z)}, \qquad \mathfrak{z}_s'(z) = \frac{\mathfrak{y}'(s, z)}{\mathfrak{y}'(\{s\} \times Z)}$$

for $z \in Z$. Since $\mathfrak{C}$ is in strongly reduced form it suffices to prove that for each $s \in W_{\mathfrak{y}}$ the assertion $\mathfrak{z}_s \sim \mathfrak{z}_s'$ holds. This is shown as follows:

$$\begin{aligned} V_{\mathfrak{C}}[\mathfrak{z}_s, p](q) &= \sum_{z \in Z} \frac{\mathfrak{y}(s, z)}{\mathfrak{y}(\{s\} \times Z)} V_{\mathfrak{C}}[z, p](q) \\ &= \frac{1}{\mathfrak{y}(\{s\} \times Z)} \sum_{z \in Z} \mathfrak{y}(s, z) \cdot V_{\mathfrak{C}}[z, p]\left(\frac{sq}{s}\right). \end{aligned}$$

Let be $s' \in W(Y)$. Then $V_{\mathfrak{C}}[z, p](sq/s') > 0$ implies $s' = s$, because from $sq/s' = \{w\} \neq \emptyset$ and $s' \neq s$ follows $l(p) = l(q) \neq l(w)$, i.e., $V_{\mathfrak{C}}[z, p](w) = 0$ since $\mathfrak{C}$ is synchronous.
Therefore, we obtain

$$\begin{aligned} V_{\mathfrak{C}}[\mathfrak{z}_s, p](q) &= \frac{1}{\mathfrak{y}(\{s\} \times Z)} \sum_{\substack{z \in Z \\ s' \in W(Y)}} \mathfrak{y}(s', z) \cdot V_{\mathfrak{C}}[z, p]\left(\frac{sq}{s'}\right) \\ &= \frac{V_{\mathfrak{C}}[\mathfrak{y}, p](sq)}{\mathfrak{y}(\{s\} \times Z)} = \frac{V_{\mathfrak{C}}[\mathfrak{y}', p](sq)}{\mathfrak{y}'(\{s\} \times Z)} = V_{\mathfrak{C}}[\mathfrak{z}_s', p](q). \end{aligned} \qquad \text{Q.E.D.}$$

3. Stochastic Operators

The last portion of the paper is devoted to an investigation of the stochastic input-output relations generated by situations within stochastic automata, i.e., of the externally observable behavior of stochastic automata, which are initially in a specified situation.

Definition 3.1. Let be X and Y nonempty sets.

1. Each function $\Phi : p \to \Phi_p$ from $W(X)$ into $\Pi_{W(Y)}$ is a stochastic operator over $[X, Y]$.

2. A stochastic operator Φ over $[X, Y]$ is said to be synchronous if $\Phi_p(q) > 0$ always implies $l(p) = l(q)$.

3. The stochastic operator Φ over $[X, Y]$ is called generated by the situation $\mathfrak{y} \in \Pi_{W(X)\times Z}$ within $\mathfrak{C} = [X, Y, Z, H]$ if $V_{\mathfrak{C}}[\mathfrak{y}, p] = \Phi_p$ for all $p \in W(X)$.

THEOREM 3.1. *If Φ is a generable stochastic operator over $[X, Y]$, then there exists a function Ψ adjoining a stochastic operator $\Psi[p, q]$ over $[X, Y]$ with each pair $[p, q] \in \{[p', q'] \mid \Phi_{p'}(q') > 0\}$ such that the equation,*

$$\Phi_{pr}(w) = \sum_{q\in W(Y),\Phi_p(q)>0} \Phi_p(q) \cdot \Psi[p, q]_r \left(\frac{w}{q}\right),$$

is satisfied for all p, $r \in W(X)$, $w \in W(Y)$.

Proof. Let be Φ generated by $\mathfrak{y}$ within $\mathfrak{C}$. If, for $p \in W(X)$, $q \in W(Y)$, $\Phi_p(q) > 0$, i.e., $V_{\mathfrak{C}}[\mathfrak{y}, p](q) > 0$, then the stochastic state $\mathfrak{C}[\mathfrak{y}, p, q]$ of $\mathfrak{C}$ is defined. Now set

$$\Psi[p, q]_r(s) =_{\mathrm{df}} V_{\mathfrak{C}}[\mathfrak{C}[\mathfrak{y}, p, q], r](s)$$

for $r \in W(X)$, $s \in W(Y)$. Obviously $\Psi[p, q]$ is a (generable) stochastic operator over $[X, Y]$. From Corollary 2.4 one obtains, that the proposed equation holds.

DEFINITION 3.2. If Φ is a stochastic operator over $[X, Y]$ and Ψ a function as in Theorem 3.1, Ψ is called a state family of Φ.

The designation "state family" was motivated by the fact, that the stochastic operators $\Psi[p, q]$ are strongly related to the states of an automaton within which Φ is generable (cf. the proof of Theorem 3.3 below). Let us first derive some consequences from the assumption, that Φ possesses a state family.

THEOREM 3.2. *If Ψ is a state family of the stochastic operator Φ over $[X, Y]$ then*

1. $\forall p \forall r \forall w(p, r \in W(X) \wedge w \in W(Y) \to \Phi_{pr}(w) \leqslant \sum_{q\in W(Y), q\sqsubseteq w} \Phi_p(q))$;
2. $\Phi_e(e) = 1 \to \Psi[e, e] = \Phi$;
3. $\forall p \forall q(p \in W(X) \wedge q \in W(Y) \wedge \Phi_p(q) > 0 \to \Psi[p, q]_e (e) =1)$;
4. $\forall p \forall q \forall r(p, r \in W(X) \wedge q \in W(Y) \to \Phi_p(q) \leqslant \sum_{s\in W(Y)} \Phi_{pr}(qs))$.

Proof. It is easily seen that the assertion 3.2.1 is a direct consequence of the definition of state family.

Let be $p \in W(X)$, $q \in W(Y)$ with $\Phi_p(q) > 0$. Then

$$\Phi_p(q) = \Phi_{ep}(q) = \sum_{s \in W(Y), \Phi_e(s) > 0} \Phi_e(s) \Psi[e, s]_p \left(\frac{q}{s}\right)$$
$$= 1 \cdot \Psi[e, e]_p(q),$$

which proves 3.2.2.

The assertion 3.2.3 is shown by induction on q. Let be $p \in W(X)$ fixed arbitrarily. For each $q \in W(Y)$ we prove

$$\Phi_p(q) > 0 \rightarrow \Psi[p, q]_e(e) = 1$$

under the assumption, that

$$\Phi_p(s) > 0 \rightarrow \Psi[p, s]_e(e) = 1$$

holds for each proper prefix s of q.

This is clear if $\Phi_p(q) = 0$. In case $\Phi_p(q) > 0$ note that

$$\Phi_p(q) = \Phi_{pe}(q) = \sum_{s \in W(Y), \Phi_p(s) > 0} \Phi_p(s) \cdot \Psi[p, s]_e \left(\frac{q}{s}\right)$$
$$= \Phi_p(q) \cdot \Psi[p, q]_e(e) + \sum_{s \in W(Y), \Phi_p(s) > 0, s \sqsubset q} \Phi_p(s) \cdot \Psi[p, s]_e \left(\frac{q}{s}\right)$$

By assumption $\Phi_p(s) > 0$ and $s \sqsubset q$ implies $\Psi[p, s]_e(e) = 1$, i.e., $\Psi[p, s]_e(q/s) = 0$ since from $s \sqsubset q$ one obtains $e \notin q/s$. Therefore, from $\Phi_p(q) > 0$ it follows $\Phi_p(q) = \Phi_p(q) \Psi[p, q]_e(e)$, i.e., $\Psi[p, q]_e(e) = 1$.

Finally we verify 3.2.4. The assertion

$$\Phi_p(q) \leqslant \sum_{s \in W(Y)} \Phi_{pr}(qs)$$

is trivial if $\Phi_p(q) = 0$. Let be $\Phi_p(q) > 0$ and $r \in W(X)$. Then

$$\sum_{s \in W(Y)} \Phi_{pr}(qs)$$
$$= \sum_{s \in W(Y)} \sum_{v \in W(Y), \Phi_p(v) > 0} \Phi_p(v) \Psi[p, v]_r \left(\frac{qs}{v}\right)$$
$$= \sum_{s \in W(Y)} \left\{ \Phi_p(q) \Psi[p, q]_r(s) + \sum_{v \in W(Y) \setminus \{q\}, \Phi_p(v) > 0} \Phi_p(v) \Psi[p, v]_r \left(\frac{qs}{v}\right) \right\}$$
$$\geqslant \Phi_p(q) \sum_{s \in W(Y)} \Psi[p, q]_r(s) = \Phi_p(q),$$

which proves 3.2.4.

Now we show that the converse of Theorem 3.1 is true, too.

THEOREM 3.3. *Each stochastic operator possessing a state family is generable.*

Proof. Let Ψ be a state family of the stochastic operator Φ over $[X, Y]$. Set $Z =_{\mathrm{df}} \{[p, q] \mid p \in W(X) \wedge q \in W(Y) \wedge \Phi_p(q) > 0\}$ and for $[p, q] \in Z$, $x \in X$, $s \in W(Y)$, $z \in Z$

$$H[[p, q], x](s, z) =_{\mathrm{df}} \begin{cases} \Psi[p, q]_x(s), & \text{if } \Psi[p, q]_x(s) > 0 \quad \text{and} \quad z = [px, qs] \\ 0, & \text{else.} \end{cases}$$

Note that if $[p, q] \in Z$ then

$$\Phi_{px}(qs) = \sum_{v \in W(Y), \Phi_p(v) > 0} \Phi_p(v) \Psi[p, v]_x \left(\frac{qs}{v}\right) \geqslant \Phi_p(q) \Psi[p, q]_x(s).$$

Therefore $[p, q] \in Z$, $\Psi[p, q]_x(s) > 0$ implies $\Phi_{px}(qs) > 0$, i.e. $[px, qs] \in Z$, which justifies the definition of H. Since

$$H[[p, q], x](W(Y) \times Z) = \sum_{s \in W(Y), \Psi[p,q]_x(s) > 0} \Psi[p, q]_x(s) = 1,$$

$\mathfrak{C} =_{\mathrm{df}} [X, Y, Z, H]$ is a stochastic automaton. Set for $s \in W(Y)$, $z \in Z$

$$\mathfrak{y}(s, z) =_{\mathrm{df}} \begin{cases} \Phi_e(s), & \text{if} \quad \Phi_e(s) > 0 \quad \text{and} \quad z = [e, s], \\ 0, & \text{else.} \end{cases}$$

Obviously $\mathfrak{y}$ is a situation of $\mathfrak{C}$.

LEMMA 3.3.a.

$$\forall p \forall q \forall z (p \in W(X) \wedge q \in W(Y) \wedge z \in Z \wedge H[\mathfrak{y}, p](q, z) > 0 \rightarrow \\ \Phi_p(q) > 0 \wedge z = [p, q]).$$

The proof is by induction on p. Initially we show that the lemma holds for $p = e$. From

$$H[\mathfrak{y}, e](q, z) = \sum_{s \in W(Y), z^* \in Z} \mathfrak{y}(s, z^*) H[z^*, e] \left(\frac{q}{s} \times \{z\}\right)$$

$$= \sum_{s \in W(Y), \Phi_e(s) > 0} \Phi_e(s) H[[e, s], e] \left(\frac{q}{s} \times \{z\}\right) > 0,$$

we obtain $\Phi_e(q)\, H[[e, q], e](e, z) > 0$ and consequently $\Phi_e(q) > 0$ and $z = [e, q]$ since $H[z', e](w, z'') > 0$ always implies $w = e$ and $z' = z''$.

Now, assuming that the lemma holds for p we shall show that it holds for px ($x \in X$). Let be p, x, q, z such that $H[\mathfrak{y}, px](q, z) > 0$. From

$$0 < H[\mathfrak{y}, px](q, z) = \sum_{s \in W(Y), z' \in Z} H[\mathfrak{y}, p](s, z')H[z', x]\left(\frac{q}{s} \times \{z\}\right)$$

one obtains that there exist a $s \in W(Y)$, $z' \in Z$ such that

(i) $H[\mathfrak{y}, p](s, z') > 0$,

(ii) $H[z', x]\left(\frac{q}{s}\right) \times \{z\}) > 0$.

By assumption from (i) follows $\Phi_p(s) > 0$ and $z' = [p, s]$, which implies by definition of H that $\Psi[p, s]_x\,(q/s) > 0$, $z = [px, q]$ and $\Phi_{px}(q) > 0$.

LEMMA 3.3.b.

$\forall p \forall q (p \in W(X) \wedge q \in W(Y) \wedge \Phi_p(q) > 0 \rightarrow H[\mathfrak{y}, p](q, [p, q]) = \Phi_p(q))$.

Again the proof is by induction on p. If $p = e$ one obtains

$$\begin{aligned} H[\mathfrak{y}, e](q, [e, q]) &= \sum_{s \in W(Y), \Phi_e(s) > 0} \Phi_e(s)H[[e, s], e]\left(\frac{q}{s} \times \{[e, q]\}\right) \\ &= \Phi_e(q)H[[e, q], e](e, [e, q]) = \Phi_e(q). \end{aligned}$$

Assuming now that the lemma holds for p we shall prove that it holds for px ($x \in X$). Let be p, x, q such that $\Phi_{px}(q) > 0$. Note that

$$H[\mathfrak{y}, px](q, [px, q]) = \sum_{w \in W(Y), z \in Z} H[\mathfrak{y}, p](w, z)H[z, x]\left(\frac{q}{w} \times \{[px, q]\}\right).$$

By Lemma 3.3.a $H[\mathfrak{y}, p](w, z) > 0$ implies $\Phi_p(w) > 0$ and $z = [p, w]$. By induction hypothesis one obtains

$$\begin{aligned} H[\mathfrak{y}, px](q, [px, q]) &= \sum_{w \in W(Y), \Phi_p(w) > 0} \Phi_p(w)H[[p, w], x]\left(\frac{q}{w} \times \{[px, q]\}\right) \\ &= \sum_{w \in W(Y), \Phi_p(w) > 0} \Phi_p(w)\Psi[p, w]_x\left(\frac{q}{w}\right) = \Phi_{px}(q). \end{aligned}$$

Now we have for each $p \in W(X)$, $q \in W(Y)$

$$\begin{aligned} V_{\mathfrak{C}}[\mathfrak{y}, p](q) &= H[\mathfrak{y}, p](\{p\} \times Z) \\ &= \begin{cases} H[\mathfrak{y}, p](q, [p, q]), & \text{if } \Phi_p(q) > 0 \\ 0, & \text{else,} \end{cases} \qquad \text{(by 3.3.a)}, \\ &= \Phi_p(q) \qquad \text{(by 3.3.b)}. \end{aligned}$$

Therefore the stochastic operator Φ is generated by the situation $\mathfrak{y}$ within the stochastic automaton $\mathfrak{C}$. Q.E.D.

COROLLARY 3.4. 1. *A stochastic operator is generable if and only if it possesses a state family.*

2. *A stochastic operator Φ is generable by a single state if and only if it possesses a state family and $\Phi_e(e) = 1$ holds.*

3. *Each generable stochastic operator can be generated within an observer-state-calculable stochastic automaton.*

If one wishes to safe states in the construction given above the following problem arises:

PROBLEM 2. Has each or at least one state family Ψ of an arbitrary generable stochastic operator Φ the property that $\Psi[p, q] = \Psi[p', q']$ and $\Psi[p, q]_x(s) > 0$ always implies $\Psi[px, qs] = \Psi[p'x, q's]$?

If this is the case one can choose the stochastic operators $\Psi[p, q]$ as states of the automaton $\mathfrak{C}$ and setting

$$H[\Psi[p, q], x](s, z) =_{\text{df}} \begin{cases} \Psi[p, q]_x(s), & \text{if } \Psi[p, q]_x(s) > 0 \text{ and } z = \Psi[px, qs] \\ 0, & \text{else,} \end{cases}$$

one obtains that the situation $\mathfrak{y}$ with

$$\mathfrak{y}(s, z) =_{\text{df}} \begin{cases} \Phi_e(s), & \text{if } z = \Psi[e, s] \text{ and } \Phi_e(s) > 0 \\ 0, & \text{else,} \end{cases}$$

generates Φ within $\mathfrak{C}$. Obviously at least one state family having the property stated can be presented, if the following problem has an affirmative solution:

PROBLEM 3. Let be $\mathfrak{C} = [X, Y, Z, H]$ a stochastic automaton, $\mathfrak{y} \in \Pi_{W(Y)\times Z}$, $p, p' \in W(X)$, $q, q' \in W(Y)$ with $V_{\mathfrak{C}}[\mathfrak{y}, p](q) > 0$ and $V_{\mathfrak{C}}[\mathfrak{y}, p'](q') > 0$. Does $\mathfrak{C}[\mathfrak{y}, p, q] \sim \mathfrak{C}[\mathfrak{y}, p', q']$ imply $\mathfrak{C}[\mathfrak{y}, px, qs] \sim \mathfrak{C}[\mathfrak{y}, p'x, q's]$ for each $x \in X$, $s \in \{w \mid V_{\mathfrak{C}}[\mathfrak{C}[\mathfrak{y}, p, q], x](s) > 0\}$?

Clearly if $\mathfrak{C}$ is synchronous this is the case since for synchronous automata from $z \sim z'$, $x \in X$, $y \in Y$ with $V_{\mathfrak{C}}[z, x](y) > 0$ it follows that $\mathfrak{C}[z, x, y] \sim \mathfrak{C}[z', x, y]$ and for $p \in W(X)$, $q \in W(Y)$ with $V_{\mathfrak{C}}[\mathfrak{y}, p](q) > 0$, $x \in X, y \in Y$ with $V_{\mathfrak{C}}[\mathfrak{C}[\mathfrak{y}, p, q], x](y) > 0$ the equation $\mathfrak{C}[\mathfrak{C}[\mathfrak{y}, p, q], x, y] = \mathfrak{C}[\mathfrak{y}, px, qy]$ is satisfied (cf. Starke (1969), p. 229).

We shall show below that each generable synchronous stochastic operator has exactly one state family and can be generated within a synchronous automaton. Therefore each state family of a generable synchronous stochastic operator has the property stated in problem 2.

The next problem again is positively solved in the synchronous case. The state family Ψ of Φ presented in the proof of 3.1 has the property that all stochastic operators $\Psi[p, q]$ are generable too. Let us call a state family generable if this is the case.

PROBLEM 4. If Φ is generable, is then each state family Ψ of Φ generable?

Let be Φ a stochastic operator over $[X, Y]$ fulfilling

$$3.2.1 \quad \forall p\, \forall r\, \forall w \left(p, r \in W(X) \wedge w \in W(Y) \rightarrow \Phi_{pr}(w) \leqslant \sum_{s \in W(Y), qw \sqsupseteq w} \Phi_p(q)\right)$$

$$3.2.4 \quad \forall p\, \forall q\, \forall r \left(p, r \in W(X) \wedge q \in W(Y) \rightarrow \Phi_p(q) \leqslant \sum_{s \in W(Y)} \Phi_{pr}(qs)\right).$$

Then Φ need not be generable, i.e., these conditions are not sufficient for generability. As an example consider the stochastic operator Φ^1 over $[\{x\}, \{a, b\}]$ with $\Phi_e{}^1(e) = 1$, $\Phi_x{}^1(a) = \Phi_x{}^1(b) = \Phi_x{}^1(aa) = \Phi_x{}^1(ab) = \frac{1}{4}$,

$$\Phi^1_{x^{n+2}}(aab^n) = \Phi^1_{x^{n+2}}(ab^{n+1}) = \Phi^1_{x^{n+2}}(b^{n+2}) = \Phi^1_{x^{n+2}}(bab^n) = \tfrac{1}{4},$$

for each $n > 0$ (whereby x^n is defined inductively by $x^0 = e$, $x^{n+1} = x^n x$). An easy computation shows that Φ^1 fulfills conditions 3.2.1 and 3.2.4. Assume that Φ^1 has a state family Ψ^1. Since $\Phi_x{}^1(b) > 0$ the stochastic operator $\Psi^1[x, a]$ is defined. One obtains on the one hand

$$\frac{1}{4} = \Phi_{xx}(ba) = \sum_{q \in W(Y), \Phi_x(q) > 0} \Phi_x{}^1(q) \Psi^1[x, q]_x \left(\frac{ba}{q}\right)$$

$$= \Phi_x{}^1(b) \Psi^1[x, b]_x(a) = \frac{1}{4} \Psi^1[x, b]_x(a),$$

i.e., $\Psi^1[x, a]_x\,(a) = 1$ and on the other hand

$$\frac{1}{4} = \Phi^1_{xx}(bb) = \Phi_x{}^1(b) \Psi^1[x, b]_x(b) = \frac{1}{4} \Psi^1[x, b]_x(b),$$

i.e., $\Psi^1[x, b]_x\,(b) = 1$, what is in contradiction to $\Psi^1[x, b]_x\,(a) = 1$.

THEOREM 3.5. *A synchronous stochastic operator Φ over $[X, Y]$ is sequential (i.e., $\Phi_{px}(\{q\} \cdot Y) = \Phi_p(q)$ holds for each $p \in W(X)$, $x \in X$, $q \in W(Y)$) if and only if it has a state family.*

Proof. For the if-part let be Ψ a state family of Φ, $p \in W(X)$, $x \in X$ and $q \in W(Y)$. If $l(p) \neq l(q)$ the equation $\Phi_{px}(\{q\} \cdot Y) = \Phi_p(q)$ is trivial, both sides are vanishing since Φ is synchronous. In case $l(p) = l(q)$ consider the equation

$$\Phi_{px}(\{q\} \cdot Y) = \sum_{y \in Y} \sum_{w \in W(Y), \Phi_p(w) > 0} \Phi_p(w) \cdot \Psi[p, w]_x \left(\frac{qy}{w}\right).$$

Now $\Phi_p(w) > 0$ implies $l(p) = l(w)$ $(= l(q))$ and, $\Psi[p, w]_x (qy/w) > 0$ implies $w \sqsubseteq qy$ from which together with $l(q) = l(w)$ follows that $q = w$. Consequently one obtains

$$\Phi_{px}(\{q\} \cdot Y) = 0 \qquad \text{if} \quad \Phi_p(q) = 0$$

and

$$\Phi_{px}(\{q\} \cdot Y) = \Phi_p(q) \sum_{y \in Y} \Psi[p, q]_x(y) = \Phi_p(q) \Psi[p, q]_x(Y),$$

if $\Phi_p(q) > 0$. Therefore it only remains to show that $\Psi[p, q]_x (Y) = 1$ if $\Phi_p(q) > 0$. Assume $\Psi[p, q]_x (Y) \neq 1$. Then there exists a tape $s \in W(Y) \setminus Y$ such that $\Psi[p, q]_x (s) > 0$, i.e., $\Phi_{px}(qs) > 0$. This is in contradiction to the assumption of Φ being synchronous since from $s \in W(Y) \setminus Y$ it follows that $l(px) \neq l(qs)$. The only-if-part directly follows from the fact that each synchronous sequential stochastic operator can be generated within a synchronous stochastic automaton (cf. Starke (1969)).

THEOREM 3.6. *There is a stochastic operator possessing a continuum of distinct state families.*

Proof. Consider the stochastic operator Φ^2 over $[\{x\}, \{a, b\}]$ with

$$\Phi_e^2(e) = 1, \qquad \Phi_x^2(a) = \Phi_x^2(aa) = \tfrac{1}{2},$$

$$\Phi^2_{x^{n+2}}(a^3 b^n) = \Phi^2_{x^{n+2}}(a^2 b^{n+1}) = \tfrac{1}{2} \qquad \text{for each} \quad n \geqslant 0$$

We prove that, for each α with $0 \leqslant \alpha \leqslant 1$, the function Ψ_α is a state family of Φ^2, where $(n, m \geqslant 0)$

$$\Psi_\alpha[e, e] = \Phi^2,$$

$$\Psi_\alpha[x, a]_{x^{n+1}}(a^2b^n) = 1 - \alpha, \qquad \Psi_\alpha[x, a]_{x^{n+1}}(ab^{n+1}) = \alpha,$$

$$\Psi_\alpha[x, aa]_{x^{n+1}}(ab^n) = \alpha, \qquad \Psi_\alpha[x, aa]_{x^{n+1}}(b^{n+1}) = 1 - \alpha,$$

$$\Psi_\alpha[x^{n+2}, a^3b^n]_{x^m}(b^m) = \Psi_\alpha[x^{n+2}, a^2b^{n+1}]_{x^m}(b^m) = 1,$$

and (for $[p, q] \in \{[x, a], [x, aa]\} \cup \{[x^{n+2}, a^3b^n], [x^{n+2}, a^2b^{n+1}] \mid n \geqslant 0\}$)

$$\Psi_\alpha[p, q]_e(e) = 1.$$

That the equation

$$\Phi^2_{pr}(w) = \sum_{q \in W(Y), \Phi_p(q) > 0} \Phi^2_p(q) \Psi_\alpha[p, q]_r \left(\frac{w}{q}\right)$$

holds is verified by simple computations and therefore left to the reader.

In connection with Theorem 3.6 the question arises under which condition a stochastic operator has one and only one state family. Moreover a criterion for the uniqueness of state family is needed. Obviously each synchronous sequential stochastic operator possesses exactly one state family; therefore, we have to look for a property which is weaker than the property of being synchronous. A property of that kind is separability.

DEFINITION 3.3. A stochastic operator Φ over $[X, Y]$ is called separable, if to each $p, r \in W(X)$, $w \in W(Y)$ with $\Phi_{pr}(w) > 0$ there corresponds exactly one prefix q of w such that $\Phi_p(q) > 0$.

COROLLARY 3.7. *Each synchronous stochastic operator is separable.*

The converse of 3.7 is not true; consider, e.g., the stochastic operator Φ with $\Phi_p(e) = 1$ (for each $p \in W(X)$) which is separable but not synchronous.

THEOREM 3.8. *Each separable stochastic operator possesses at most one state family.*

Proof. Let be Ψ_1, Ψ_2 state families of the separable stochastic operator Φ

over $[X, Y]$ and $p, r \in W(X)$, $q, s \in W(Y)$ with $\Phi_p(q) > 0$. Then for $i \in \{1, 2\}$ one obtains:

(i) In case $\Phi_{pr}(qs) > 0$,

$$\begin{aligned}\Psi_i[p, q]_r(s) &= \frac{\Phi_p(q)}{\Phi_p(q)} \Psi_i[p, q]_r(s)\\ &= \frac{1}{\Phi_p(q)} \sum_{v \in W(Y), \Phi_p(v) > 0} \Phi_p(v) \cdot \Psi_i[p, v]_r \left(\frac{qs}{v}\right)\\ &= \frac{\Phi_{pr}(qs)}{\Phi_p(q)},\end{aligned}$$

since q is the only prefix v of qs such that $\Phi_p(v) > 0$.

(ii) In case $\Phi_{pr}(qs) = 0$,

$$0 = \sum_{v \in W(Y), \Phi_p(v) > 0} \Phi_p(v) \Psi_i[p, v]_r \left(\frac{qs}{v}\right) \geqslant \Phi_p(q) \Psi_i[p, q]_r(s) \geqslant 0,$$

i.e.,

$$\Psi_i[p, q]_r(s) = 0 = \frac{\Phi_{pr}(qs)}{\Phi_p(q)}.$$

COROLLARY 3.9. *If Φ is separable and Ψ a state family of Φ, then*

1. $$\forall p\, \forall q\, \forall r\, \forall s \left(p, r \in W(X) \wedge q, s \in W(Y) \wedge \Phi_p(q) > 0 \right.$$ $$\left. \rightarrow \Psi[p, q]_r(s) = \frac{\Phi_{pr}(qs)}{\Phi_p(q)}\right)$$

2. $$\forall p\, \forall q\, \forall r \left(p, r \in W(X) \wedge q \in W(Y) \wedge \Phi_p(q) > 0 \right.$$ $$\left. \rightarrow \sum_{s \in W(Y)} \Phi_{pr}(qs) = \Phi_p(q)\right).$$

The assertion 3.9.2 is a direct consequence from $\sum_{s \in W(Y)} \Psi[p, q]_r(s) = 1$ and 3.9.1. From Theorem 3.8 it follows that Problem 4 stated above is positively solved in the case that Φ is separable.

Let us now reconsider Problem 2 in case that Φ is separable. Let be Ψ an arbitrary state family of Ψ, $\Psi[p, q] = \Psi[p', q']$ and $\Psi[p, q]_x(s) > 0$. Then one obtains from 3.9.1

$$\Phi_{px}(qs) = \Phi_p(q)\,\Psi[p, q]_x(s) > 0,$$

$$\Phi_{p'x}(q's) = \Phi_{p'}(q')\,\Psi[p', q']_x(s) = \Phi_{p'}(q')\,\Psi[p, q]_x(s) > 0,$$

and therefore for each $r \in W(X)$, $v \in W(Y)$

$$\Psi[px, qs]_r(v) = \frac{\Phi_{pxr}(qsv)}{\Phi_{px}(qs)} = \frac{\Phi_p(q)\Psi[p, q]_{xr}(sv)}{\Phi_p(q)\Psi[p, q]_x(s)}$$

$$= \frac{\Psi[p', q']_{xr}(sv)}{\Psi[p', q']_x(s)} = \Psi[p'x, q's]_r(v),$$

i.e., $\Psi[px, qs] = \Psi[p'x, q's]$. Consequently, if Φ is separable each state family of Φ has the property mentioned in Problem 2.

Naturally, there are separable stochastic operators possessing no state family. Each stochastic operator being synchronous but not sequential is of that kind.

THEOREM 3.10. *Separability is not a necessary condition for the uniqueness of state family.*

Proof. Consider the stochastic operator Φ^3 over $[\{x\}, \{a, b\}]$ with $\Phi_e^3(e) = 1$, $\Phi_x^3(a) = \Phi_x^3(aa) = \frac{1}{2}$, $\Phi^3_{x^{n+2}}(a^3b^n) = 1$ for each $n \geqslant 0$. Obviously Φ^3 is not separable (consider $p = r = x$, $w = a^3$) but Φ^3 has exactly one state family. This is seen as follows. Each state family Ψ of Φ^3 is defined on the set

$$Q = \{[e, e], [x, a], [x, aa]\} \cup \{[x^{n+2}, a^3b^n] \mid n \geqslant 0\}.$$

Since $\Phi_e^3(e) = 1$ by 3.2.2 the value $\Psi[e, e]$ is unique ($\Psi[e, e] = \Phi^3$). From 3.2.3 one obtains $\Psi[p, q]_e(e) = 1$ for each $[p, q] \in Q$. For each state family Ψ of Φ^3, $n \geqslant 0$ one obtains

$$1 = \Phi^3_{x^{n+2}}(a^3b^n) = \sum_{q \in W(Y), \Phi_x^3(q) > 0} \Phi_x^3(q)\Psi[x, q]_{x^{n+1}}\left(\frac{a^3b^n}{q}\right)$$

$$= \frac{1}{2}\Psi[x, a]_{x^{n+1}}(a^2b^n) + \frac{1}{2}\Psi[x, aa]_{x^{n+1}}(ab^n).$$

Consequently,

$$\Psi[x, a]_{x^{n+1}}(a^2b^n) = 1$$

and

$$\Psi[x, aa]_{x^{n+1}}(ab^n) = 1,$$

i.e., the values $\Psi[x, a]$, $\Psi[x, aa]$ of Ψ are unique. A similar argument shows that the values $\Psi[x^{n+2}, a^3b^n]$ of Ψ are unique too.

THEOREM 3.11. *Each generable stochastic operator fulfilling condition 3.9.2 is separable.*

Proof. Let be Φ a generable stochastic operator over $[X, Y]$ fulfilling 3.9.2 and Ψ a state family of Φ. From 3.9.2 one obtains for $p, r \in W(X)$, $q \in W(Y)$ with $\Phi_p(q) > 0$.

$$\begin{aligned} 0 &= \Phi_p(q) - \sum_{s \in W(Y)} \Phi_{pr}(qs) \\ &= \Phi_p(q) - \sum_{s \in W(Y)} \sum_{v \in W(Y), \Phi_p(v) > 0} \Phi_p(v)\Psi[p, v]_r\left(\frac{qs}{v}\right) \\ &= \Phi_p(q) - \sum_{s \in W(Y)} \Big[\Phi_p(q)\Psi[p, q]_r(s) + \sum_{v \in W(Y)\setminus\{q\}, \Phi_p(v) > 0} \Phi_p(v)\Psi[p, v]_r\left(\frac{qs}{v}\right)\Big] \\ &= \sum_{s \in W(Y)} \sum_{v \in W(Y)\setminus\{q\}, \Phi_p(v) > 0} \Phi_p(v)\Psi[p, v]_r\left(\frac{qs}{v}\right). \end{aligned}$$

We have to show that

$$\Phi_{pr}(w) > 0 \wedge \Phi_p(q) > 0 \wedge \Phi_p(q') > 0 \wedge q, q' \sqsubseteq w$$

always implies $q = q'$. Assume that $q \neq q'$. Since $q, q' \sqsubseteq w$ without loss of generality one can assume moreover that q is a proper prefix of q'. From this now follows

$$\begin{aligned} 0 &= \sum_{s \in W(Y)} \sum_{v \in W(Y)\setminus\{q\}, \Phi_p(v) > 0} \Phi_p(v)\Psi[p, v]_r\left(\frac{qs}{v}\right) \\ &\geqslant \sum_{s \in W(Y)} \Phi_p(q')\Psi[p, q']_r\left(\frac{qs}{q'}\right) \geqslant \Phi_p(q') \cdot \sum_{s \in (q'/q)W(Y)} \Psi[p, q']_r\left(\frac{qs}{q'}\right) \\ &= \Phi_p(q')\Psi[p, q']_r(W(Y)) = \Phi_p(q'), \end{aligned}$$

i.e., one obtains $\Phi_p(q') \leqslant 0$ in contradiction to $\Phi_p(q') > 0$. Consequently, Φ is separable.

COROLLARY 3.12. *If Φ is a generable stochastic operator over $[X, Y]$ the following three conditions are pairwise equivalent*:

(a) *Φ is separable.*

(b) *Φ has exactly one state family Ψ and for each $p, r \in W(X), q, s \in W(Y)$ with $\Phi_p(q) > 0$ the equation*

$$\Psi[p, q]_r(s) = \frac{\Phi_{pr}(qs)}{\Phi_p(q)}$$

is satisfied.

(c) $\forall p \forall r \forall q(p, r \in W(X) \wedge q \in W(Y) \wedge \Phi_p(q) > 0$
$\rightarrow \Phi_p(q) = \sum_{s \in W(Y)} \Phi_{pr}(qs))$.

Let be Φ a separable stochastic operator over $[X, Y]$ fulfilling condition 3.9.2 (i.e., 3.12, (c)). For $p, r \in W(X)$, $q, s \in W(Y)$ with $\Phi_p(q) > 0$ let be

$$\Psi[p, q]_r(s) =_{\mathrm{df}} \frac{\Phi_{pr}(qs)}{\Phi_p(q)}.$$

From 3.9.2 one obtains that $\Psi[p, q]$ is a discrete probability measure over $W(Y)$. Note that for each $p, r \in W(X)$, $w \in W(Y)$ by definition of Ψ

$$\begin{aligned} \sum_{q \in W(Y), \Phi_p(q) > 0} \Phi_p(q) \Psi[p, q]_r \left(\frac{w}{q}\right) &= \sum_{q \in W(Y), \Phi_p(q) > 0} \Phi_{pr}(w) \\ &= \operatorname{card}(\{q \mid q \sqsubseteq w \wedge \Phi_p(q) > 0\}) \Phi_{pr}(w) \\ &= \Phi_{pr}(w), \end{aligned}$$

since Φ is separable, i.e., from $\Phi_{pr}(w) > 0$ it follows that

$$\operatorname{card}(\{q \mid q \sqsubseteq w \wedge \Phi_p(q) > 0\}) = 1.$$

Consequently, Ψ is a state family of Φ, i.e., Φ is generable. From this and the preceding results we obtain

THEOREM 3.13. *Let be Φ an arbitrary stochastic operator over $[X, Y]$.*

1. *If Φ is separable then Φ has a state family if and only if Φ fulfills condition* 3.9.2.

2. *If Φ possesses a state family then Φ is separable if and only if Φ fulfills condition* 3.9.2.

3. *If Φ fulfills condition* 3.9.2 *then Φ is separable if and only if Φ has a state family.*

Reconsider the stochastic operator Φ^2 over $[\{x\}, \{a, b\}]$ which proved Theorem 3.6. One easily verifies that the state families Ψ_0, Ψ_1 of Φ ($\alpha = 0$, $\alpha = 1$, respectively) have the property:

If $\Phi_p(q) \cdot \Phi_p(q') > 0$, $\Psi[p, q]_r(s) \cdot \Psi[p, q']_r(s') > 0$

and $qs = q's'$ then $q = q'$ and $s = s'$.

Obviously this is a separability property of Φ in relation to one of its state families Ψ. It is clear that each separable stochastic operator possessing a state family has that property.

DEFINITION 3.4. Let be Ψ a state family of the stochastic operator Φ over $[X, Y]$. Φ is said to be separable in relation to Ψ if for all $p, r \in W(X)$, $q, q', s, s' \in W(Y)$ from $\Phi_p(q)\, \Phi_p(q') > 0$, $\Psi[p, q]_r(s)\, \Psi[p, q']_r(s') > 0$ and $qs = q's'$ follows $q = q'$.

COROLLARY 3.14. *Let be Φ a stochastic operator over $[X, Y]$.*

1. *Φ is separable in relation to Ψ if and only if to each $p, r \in W(X)$, $w \in W(Y)$ with $\Phi_{pr}(w) > 0$ there corresponds one and only one dissection $w = qs$ such that $\Phi_p(q) > 0$ and $\Psi[p, q]_r(s) > 0$.*

2. *If Φ is separable and Φ has a state family Ψ then Φ is separable in relation to Ψ.*

Clearly there are stochastic operators (e.g., Φ^2) being separable in relation to one of their state families but not separable.

THEOREM 3.15. *Let Ψ be a state family of the stochastic operator Φ over $[X, Y]$. Then Φ is separable in relation to Ψ if and only if for all $p, r \in W(X)$, $q \in W(Y)$ with $\Phi_p(q) > 0$ the equation*

$$\Phi_p(q) = \sum_{s \in W(Y),\, \Psi[p,q]_r(s) > 0} \Phi_{pr}(qs)$$

holds.

Proof. For the if-part let be $\Phi_p(q)\, \Phi_p(q') > 0$, $\Psi[p, q]_r(s) \cdot \Psi[p, q']_r(s') > 0$ and $qs = q's'$. Assume that $q \neq q'$, i.e., without loss of generality, $q \sqsubset q'$. An argument similiar to that one used in the proof of 3.11 shows that from the assumed equation follows:

$$0 = \sum_{v \in W(Y), \Psi[p,q']_r(v)>0} \quad \sum_{v' \in W(Y)\setminus\{q'\}, \Phi_p(v')>0} \Phi_p(v')\Psi[p, v']_r \left(\frac{q'v}{v'}\right).$$

Now consider the case $v = s'$, $v' = q$. One obtains

$$0 = \Phi_p(q)\Psi[p, q]_r \left(\frac{q's'}{q}\right) = \Phi_p(q)\Psi[p, q]_r \left(\frac{qs}{q}\right)$$
$$= \Phi_p(q)\Psi[p, q]_r(s) > 0,$$

i.e., the assumption $q \neq q'$ leads to a contradiction. For the only-if-part note that if Φ is separable in relation to Ψ then for $p, r \in W(X)$, $q, s \in W(Y)$ holds

$$\Phi_p(q) > 0 \wedge \Psi[p, q]_r(s) > 0 \rightarrow \Phi_{pr}(qs) = \Phi_p(q)\,\Psi[p, q]_r(s).$$

From this one obtains (if $\Phi_p(q) > 0$)

$$\sum_{s \in W(Y), \Psi[p,q]_r(s)>0} \Phi_{pr}(qs) = \Phi_p(q) \sum_{s \in W(Y), \Psi[p,q]_r(s)>0} \Psi[p, q]_r(s)$$
$$= \Phi_p(q).$$

THEOREM 3.16. *If the stochastic operator Φ over $[X, Y]$ has at least two distinct state families Ψ_1, Ψ_2 then Φ possesses a state family $\tilde{\Psi}$ such that Φ is not separable in relation to $\tilde{\Psi}$.*

Proof. For $p, r \in W(X)$, $q, s \in W(Y)$ with $\Phi_p(q) > 0$ let be

$$\tilde{\Psi}[p, q]_r(s) =_{\mathrm{df}} \tfrac{1}{2}\Psi_1[p, q]_r(s) + \tfrac{1}{2}\Psi_2[p, q]_r(s).$$

It is easy to verify that $\tilde{\Psi}$ is a state family of Φ. Since Ψ_1, Ψ_2 are distinct there are tapes $p, r \in W(X)$, $q, s \in W(Y)$ such that

$$\Phi_p(q) > 0,\ \Psi_1[p, q]_r(s) \neq \Psi_2[p, q]_r(s).$$

From

$$\sum_{v \in W(Y), \Phi_p(v)>0} \Phi_p(v)\Psi_1[p, v]_r \left(\frac{qs}{v}\right)$$
$$= \Phi_{pr}(qs) = \sum_{v \in W(Y), \Phi_p(v)>0} \Phi_p(v)\Psi_2[p, v]_r \left(\frac{qs}{v}\right)$$

it follows now that there exists at least one tape $q' \in W(Y)$ fulfilling

$$q \neq q', \qquad \Phi_p(q') > 0, \qquad \Psi_1[p, q']_r \left(\frac{qs}{q'}\right) \neq \Psi_2[p, q']_r \left(\frac{qs}{q'}\right).$$

Therefore there is a tape s' with $qs = q's'$ and one obtains

$$\tilde{\Psi}[p, q]_r (s) = \tfrac{1}{2}\Psi_1[p, q]_r (s) + \tfrac{1}{2}\Psi_2[p, q]_r (s) > 0,$$
$$\tilde{\Psi}[p, q']_r (s') = \tfrac{1}{2}\Psi_1[p, q']_r (s') + \tfrac{1}{2}\Psi_2[p, q']_r (s') > 0,$$

i.e. Φ is not separable in relation to $\tilde{\Psi}$.

If one wants to prove the converse of 3.14.2 the following problem arises.

PROBLEM 5. Do stochastic operators exist which have exactly one state family Ψ and which are separable in relation to Ψ but not separable?

If this is not the case one can easily see that a stochastic operator is separable if and only if it is separable in relation to each of its state-families.

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